Sampling a Neighbor in High Dimensions

Who is the fairest of them all?

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Joint work with

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UIUC

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BARC and University of Copenhagen

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University of Padova

Main motivation in the context of Fairness

Goal of fairness: Remove or minimize the harm caused by the algorithms

- Bias in data
- Bias in the data structures that handle it

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- Selection bias, not introduce it
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- Similarity search (Near Neighbor problem)

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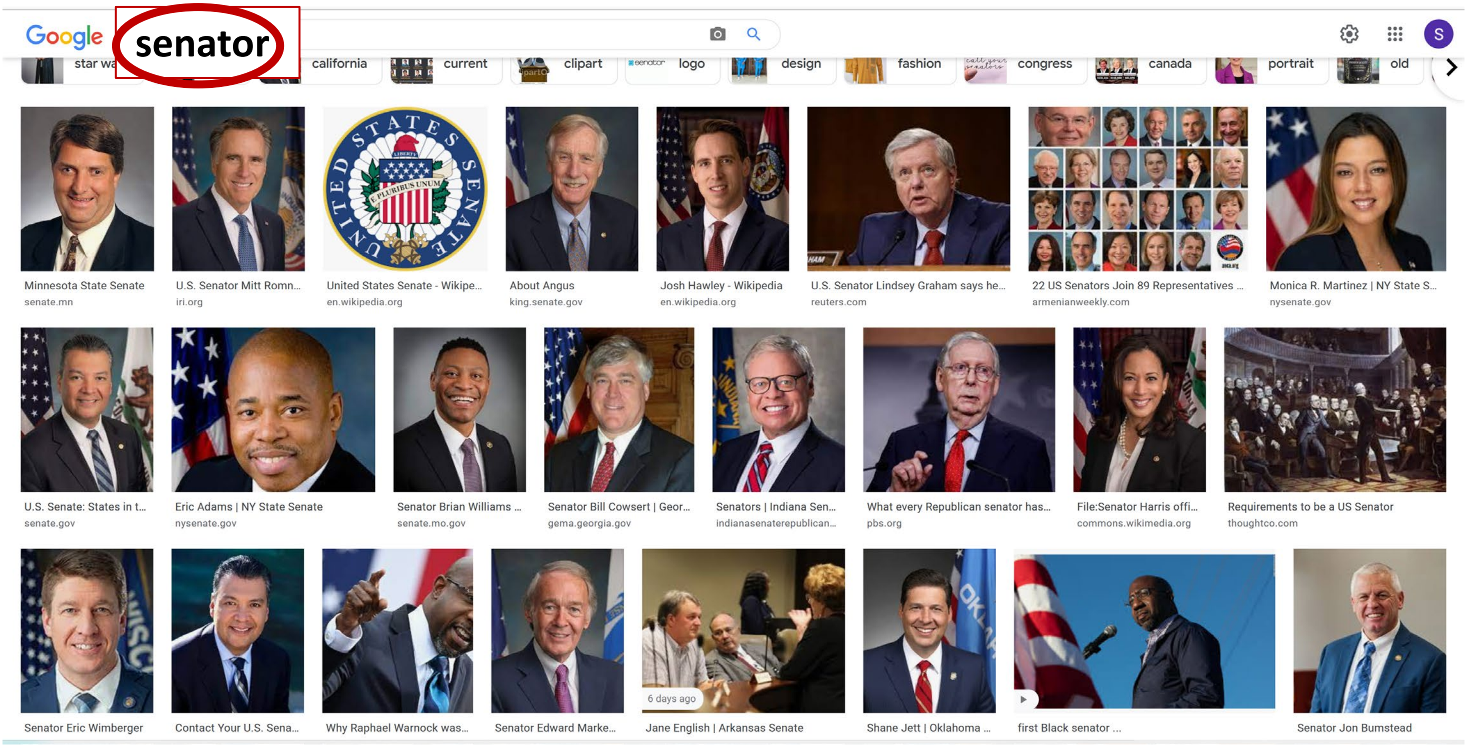
➢ No unique definition of fairness, e.g.

- Group fairness: demographics of the population are preserved in the outcome
- Individual fairness: treat individuals with similar conditions similarly, equal opportunity

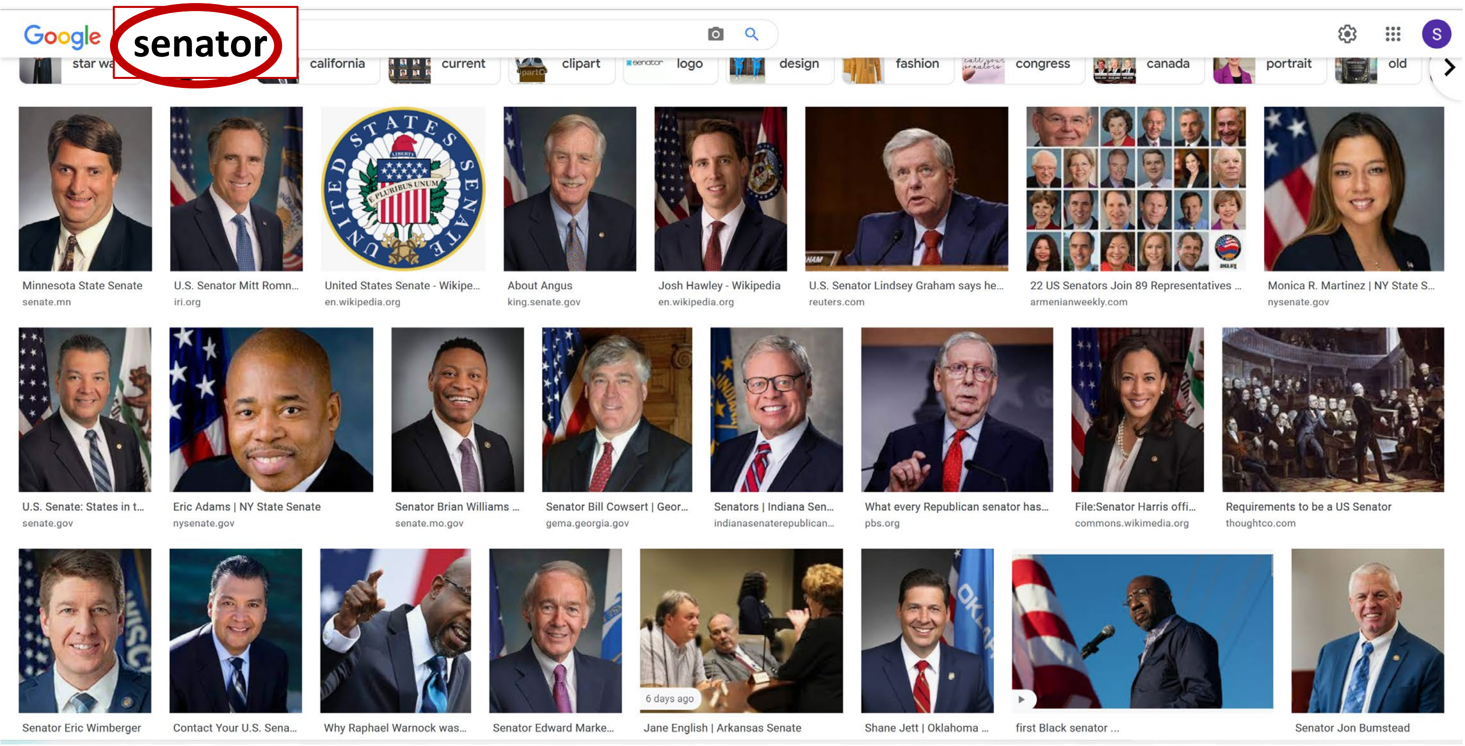
Individual Fairness in Searching

- 27% of senators are women



Individual Fairness in Searching

- 27% of senators are women
- Searching for job applicants (e.g. LinkedIn suggestions)

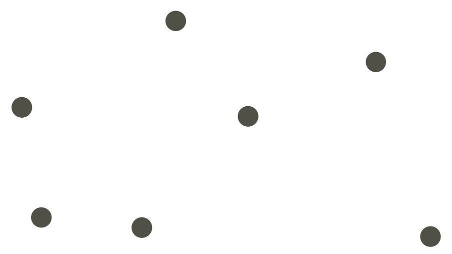


Plan for the talk

- Nearest neighbor
- Sampling version/ fair version
- Applications
- Algorithms
  - Basic Algorithm
  - Improving the dependence on ϵ
  - Handling Outliers
  - Improving the dependence on the neighborhood

Near Neighbor

Dataset of n points P in a metric space, e.g. $\mathbb{R}^d$, and a parameter r



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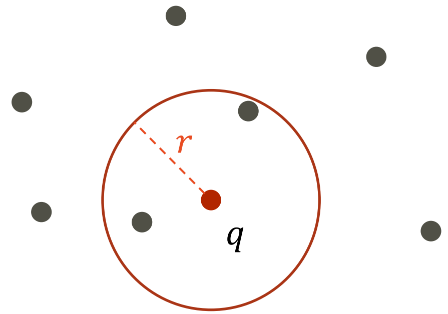

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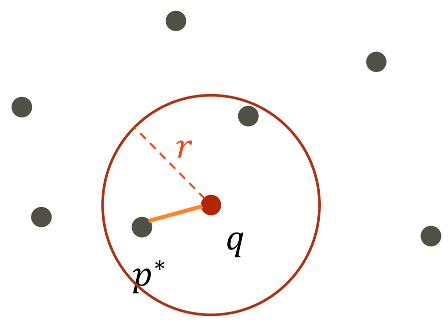

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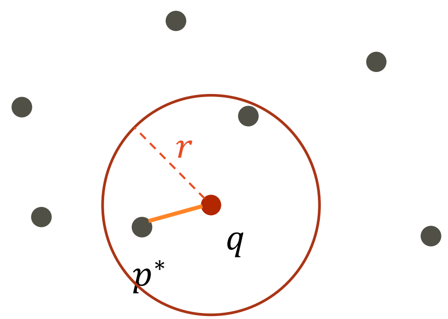

Goal:

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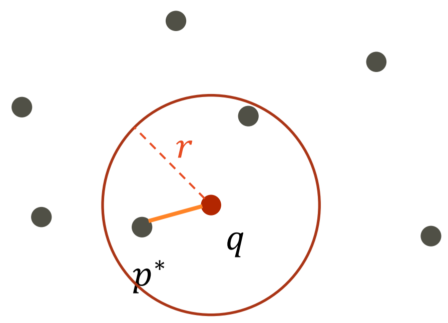

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All existing algorithms for this problem

- Either space or query time depending exponentially on d
- Or assume certain properties about the data, e.g., bounded intrinsic dimension

Approximate Near Neighbor

Dataset of n points P in a metric space, e.g. $\mathbb{R}^d$, and a parameter r

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Goal:

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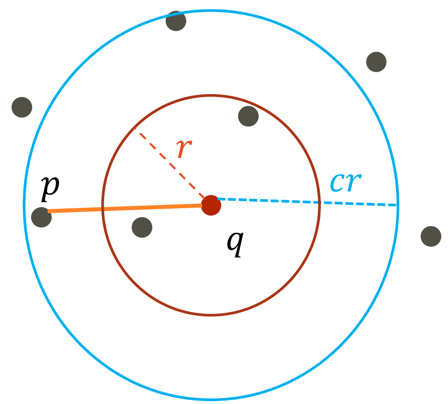
- Do it in sub-linear time and small space
- Approximate Near Neighbor
  - Report a point in distance cr for $c > 1$



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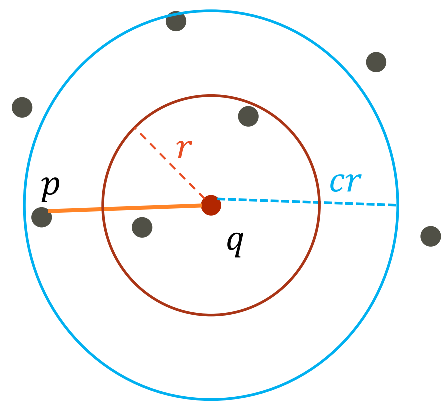

Goal:

- Find a point p^* in the r-neighborhood
- Do it in sub-linear time and small space
- Approximate Near Neighbor
  - Report a point in distance cr for $c > 1$
  - For Hamming (and Manhattan) query time is $n^{O(1/c)}$ [IM98]
  - and for Euclidean it is $n^{O(\frac{1}{c^2})}$ [AI08]

Fair Near Neighbor

Report one of the neighbors uniformly at random

- ❑ Individual fairness: every neighbor has the same chance of being reported.
- ❑ Remove the bias inherent in the NN data structure (also for the downstream tasks)

- ➢ Fair Near Neighbor as a **NN sampling problem**:
  - Sample a point in the neighborhood of the query uniformly at random

Beyond Fairness: When random nearby-by is better than the nearest

- ❑ Robustness: input is noisy, and the closest point might be an unrepresentative outlier

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- ❑ KNN-Classification

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- ➢ small values of k, are not robust
- ➢ large values are not time efficient

➢Instead: sample a few points in the neighborhood and assign the label based on the majority of sampled points

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- ❑ Filtered Searching

Applications beyond Fairness: Filtered Searching

- Apply filters on top of our search.
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  - Searches for “shoes”
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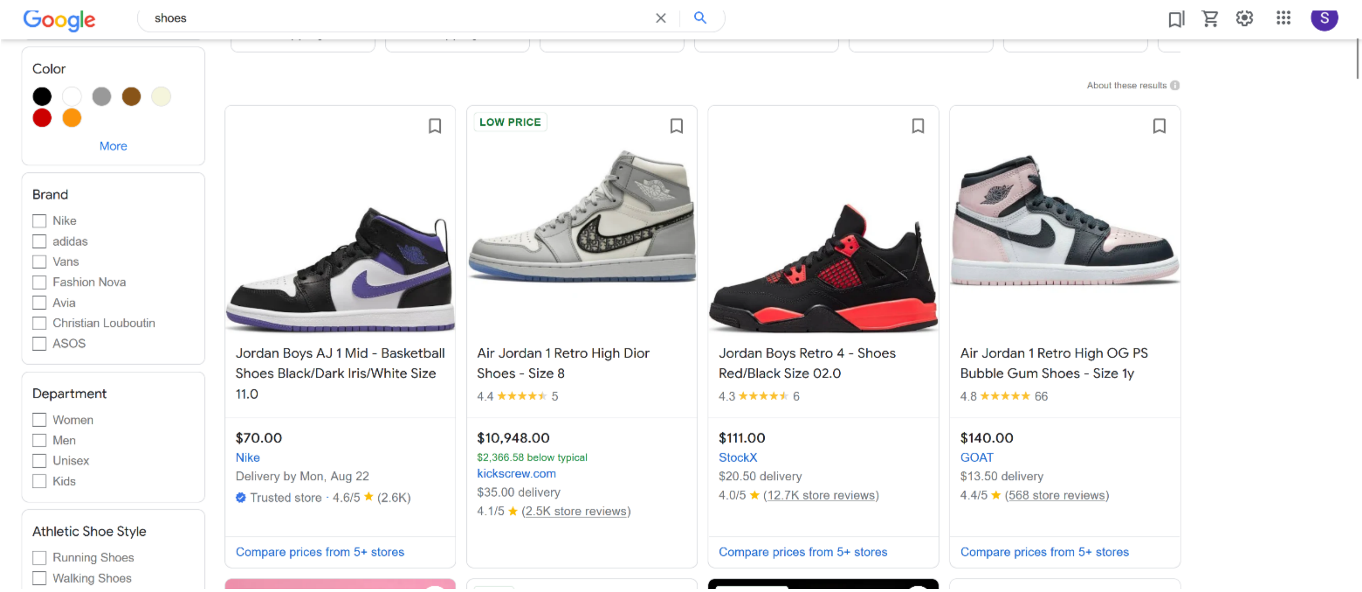

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- E.g. in a shopping scenario, person looking for “blue” shoes
  - Searches for “shoes”
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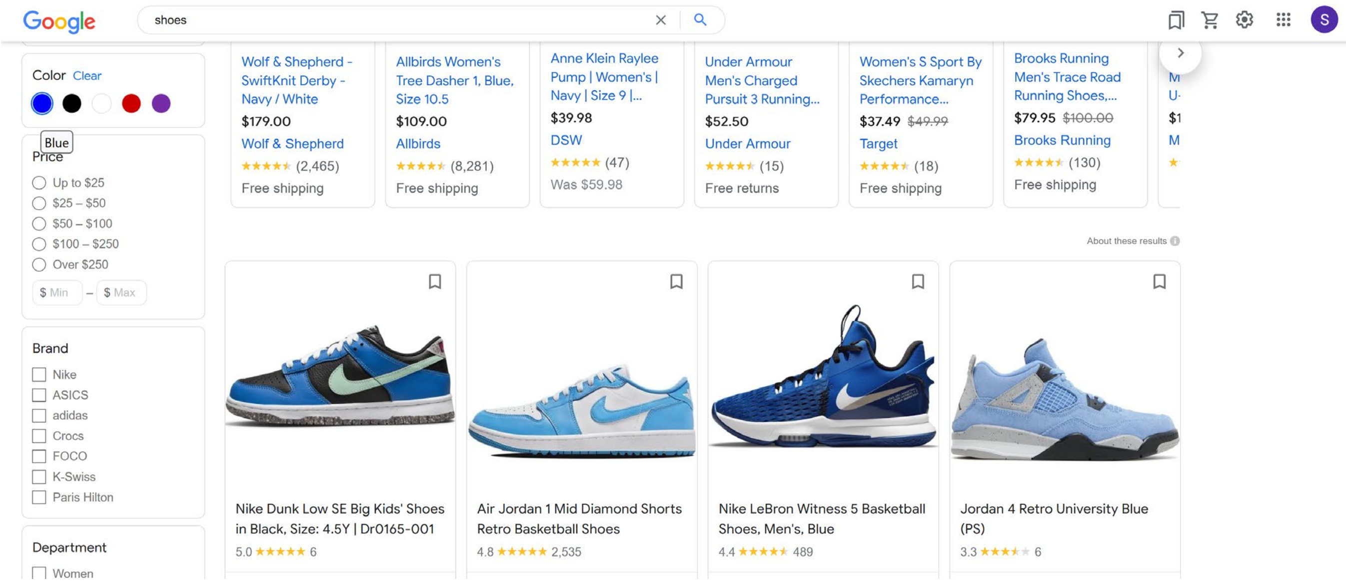
- If the desired property is common in the neighborhood:
  - Retrieve random shoes until blue shoes are found.
  - Can be combined with a different procedure for rare filters



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- ❑ Filtered Searching
- ❑ Anonymizing the data
- ❑ Diversifying the output (e.g. in a recommendation system)

Problem formulation and our results

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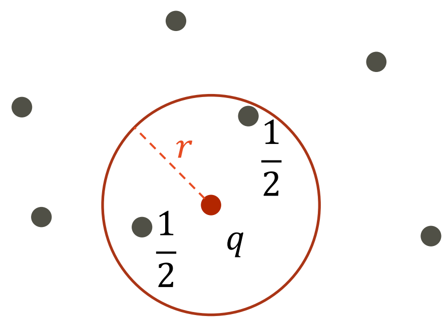

Goal:

- Return each point p in the neighborhood of q with uniform probability
- Do it in sub-linear time and small space

Approximately Fair Near Neighbor

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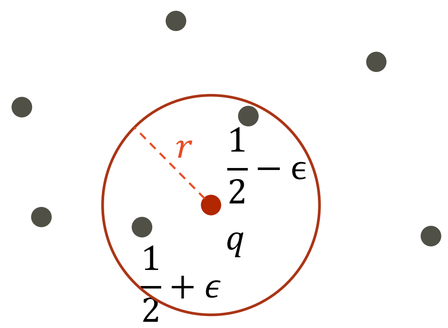

Goal of Approximately Fair NN

– Any point p in $N(q, r)$ is reported with "almost uniform" probability, i.e., $\lambda_q(p)$ where

$$\frac{1}{(1+\epsilon)|N(q,r)|} \leq \lambda_q(p) \leq \frac{(1+\epsilon)}{|N(q,r)|}$$

Further notes

Need Independence

- Need a Fresh Sample each time, i.e., require independence between queries:

$$\mathbf{Pr}\left[\boldsymbol{out}_{\boldsymbol{i},\boldsymbol{q}_{\boldsymbol{i}}} = \boldsymbol{p} \middle| \boldsymbol{out}_{\boldsymbol{i}-\mathbf{1},\boldsymbol{q}_{\boldsymbol{i}-\mathbf{1}}} = \boldsymbol{p}_{\boldsymbol{i}-\mathbf{1}}, \dots, \boldsymbol{out}_{\mathbf{1},\boldsymbol{q}_{\mathbf{1}}} = \boldsymbol{p}_{\mathbf{1}}\right] \approx \frac{\mathbf{1}}{|\boldsymbol{N}(\boldsymbol{q},\boldsymbol{r})|}$$

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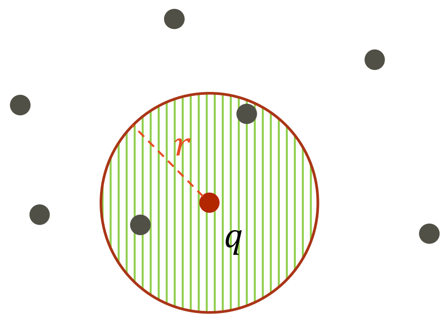
Pior Work

- In low dimensions, “Independent Range Sampling” [Xiaocheng Hu, Miao Qiao, and Yufei Tao.]
  - Exponential dependence on dim runtime

Results on $(1+\epsilon)$-Approximate Fair NN

Domain	Space	Query
Exact Neighborhood $N(q,r)$	$O(S_{ANN})$	$\tilde{O}(T_{ANN} + \frac{\lvert N(q,cr)\rvert}{\lvert N(q,r)\rvert})$

- S_{ANN} and T_{ANN} are the space and query time of standard ANN



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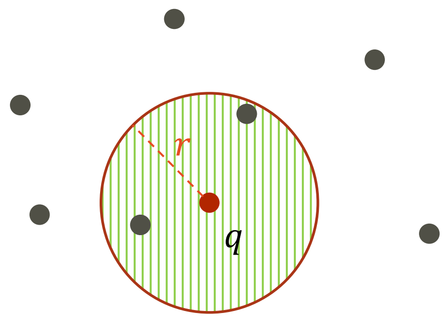

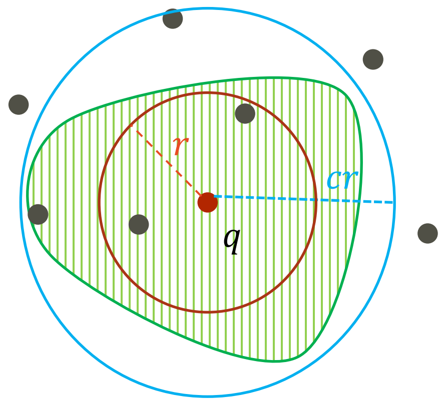

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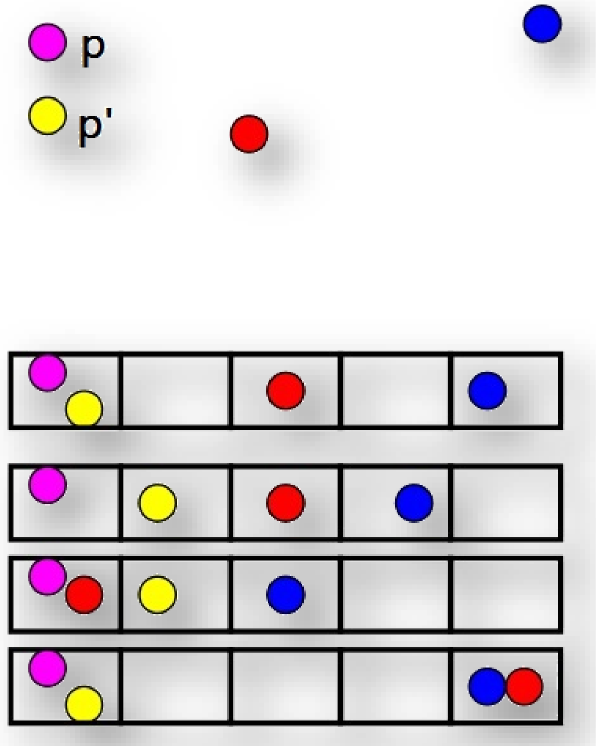
- Our approach solves a more general problem
- Experiments (Naïve randomization of ANN is not fair)

Locality Sensitive Hashing (LSH) [Indyk, Motwani'98]

One of the main approaches to solve the Nearest Neighbor problems

Locality Sensitive Hashing (LSH)

Hashing scheme s.t. close points have higher probability of collision than far points



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Hash functions: $g_1, \dots, g_L$

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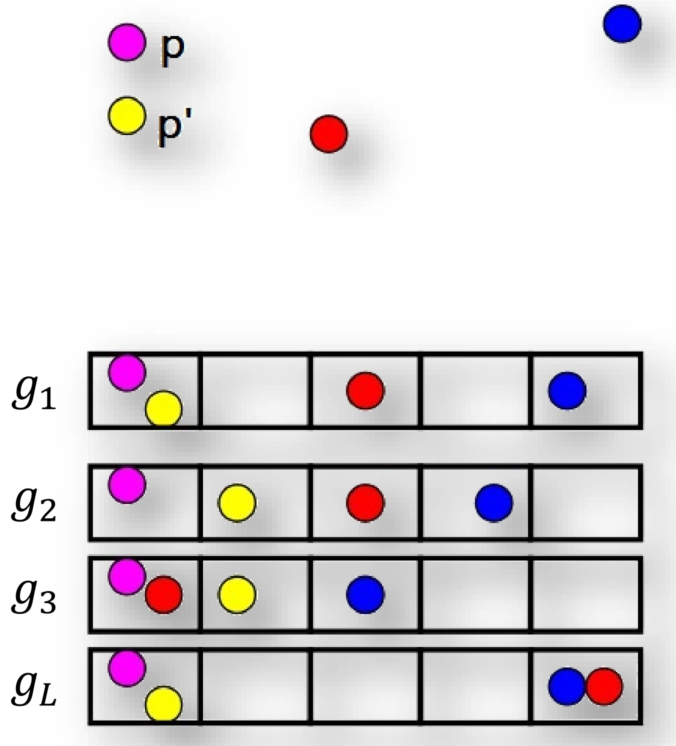

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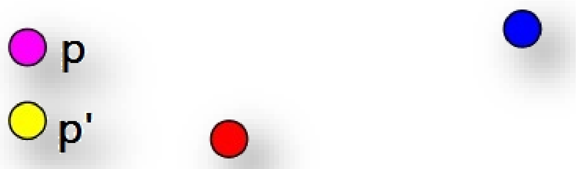
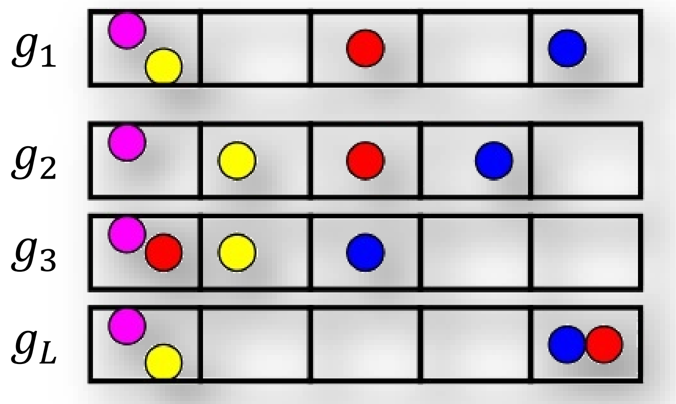


If $||p - p'|| \leq r$, they collide w.p. $\geq P_{high}$

If $||p - p'|| \geq cr$, they collide w.p. $\leq P_{low}$

For $P_{high} \geq P_{low}$



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Retrieval: [Indyk, Motwani'98]

- The union of the query buckets is roughly the neighborhood of q

- $\bigcup_i B_i\left(g_{i(q)}\right)$ is roughly the neighborhood
  - Contains all points within distance r
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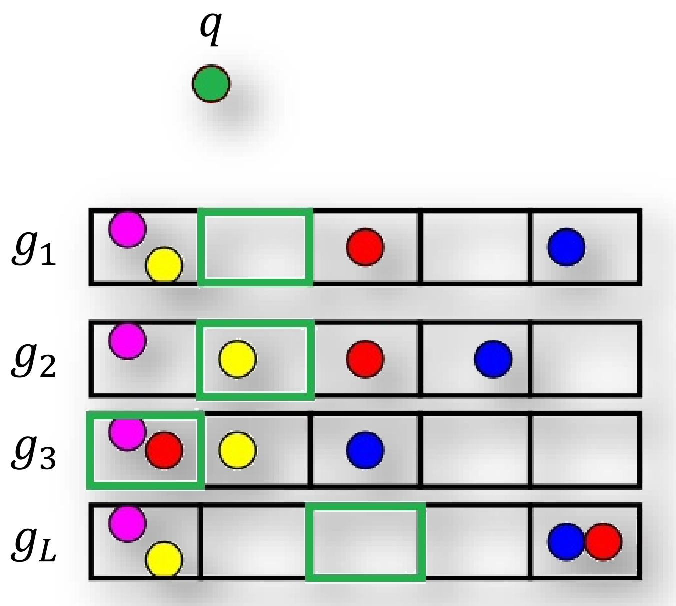

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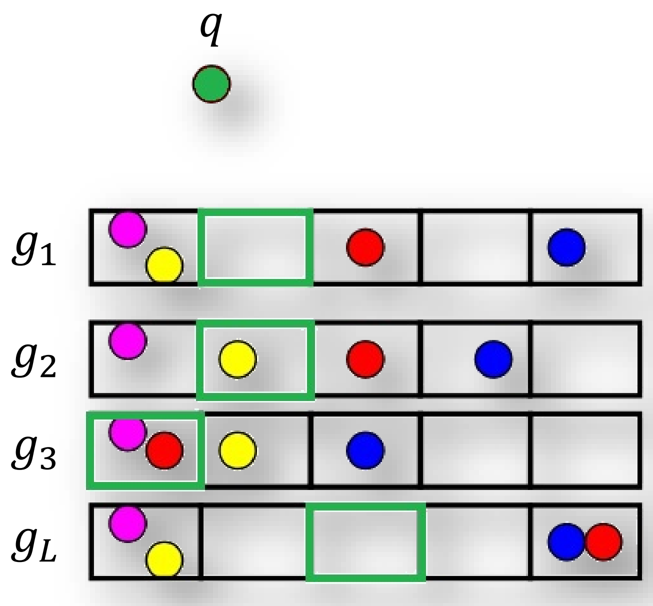

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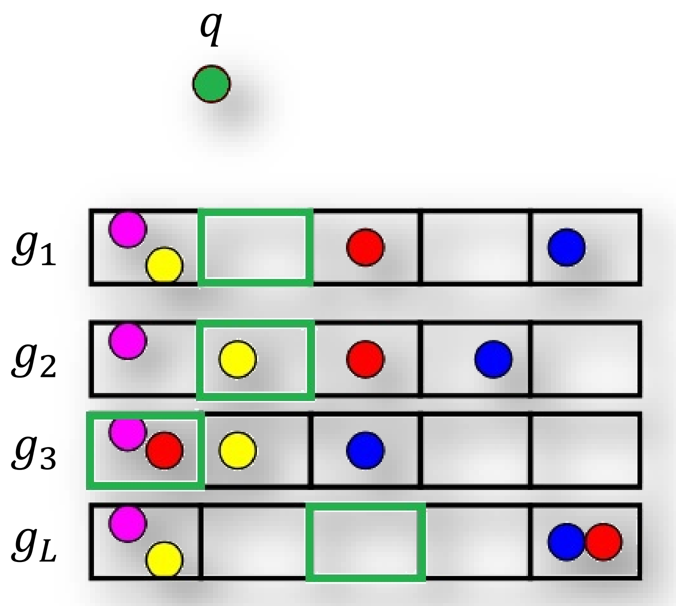
  - Collecting all points might take $O(n)$ time



A more general problem

Sampling from a sub-collection of Sets

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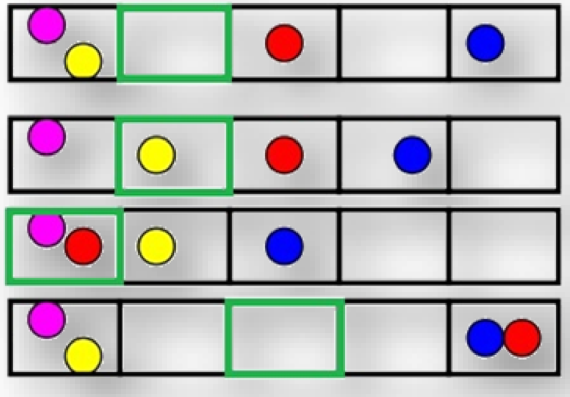

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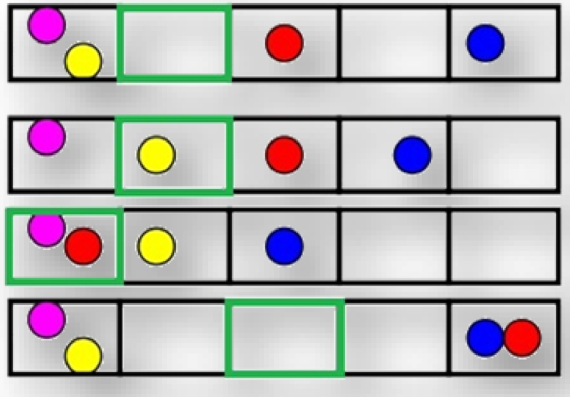

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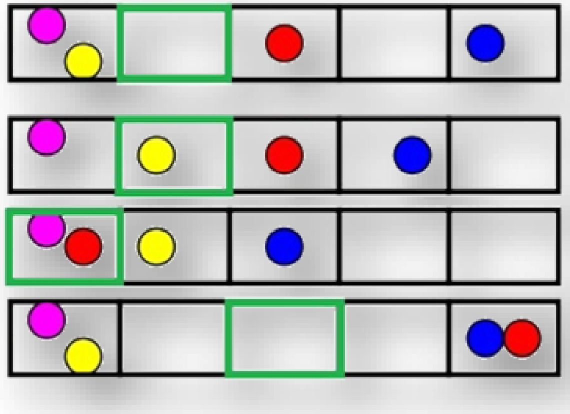

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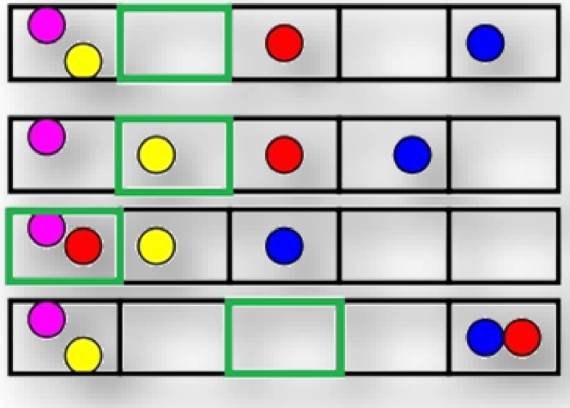

Other applications:

- Sampling from neighbors of a subset of vertices in a graph
- Uniform sampling for range searching

- Nearest neighbor
- Sampling version/ fair version
- Applications
- Algorithms
  - Basic Algorithm
  - Improving the dependence on ϵ
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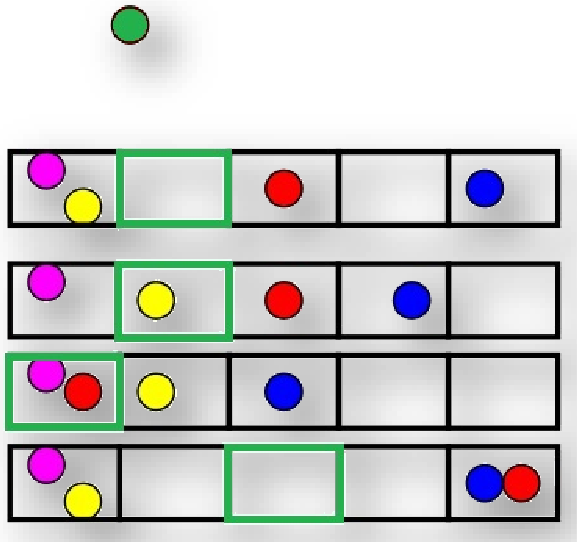
Basic Algorithm

Algorithm

How to output a random neighbor from $\cup\mathcal{G} = \cup_{F\in\mathcal{G}} F$

1. Choose a set $F \in \mathcal{G}$ w.p. $\propto |F|$
2. Choose a uniformly random point in F



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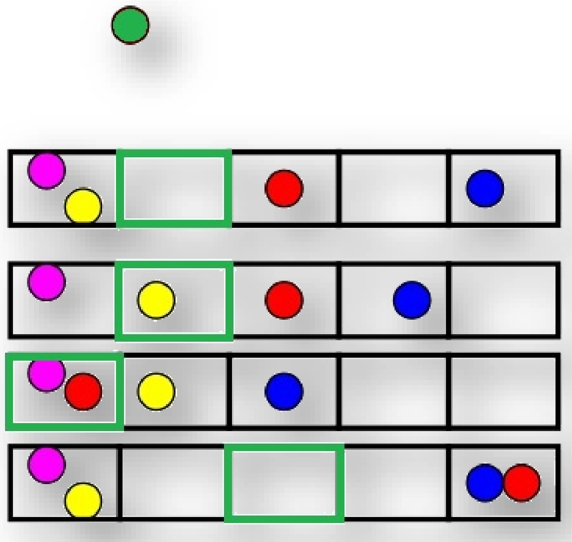
Number of sets in $\mathcal{G}$ that p appears in



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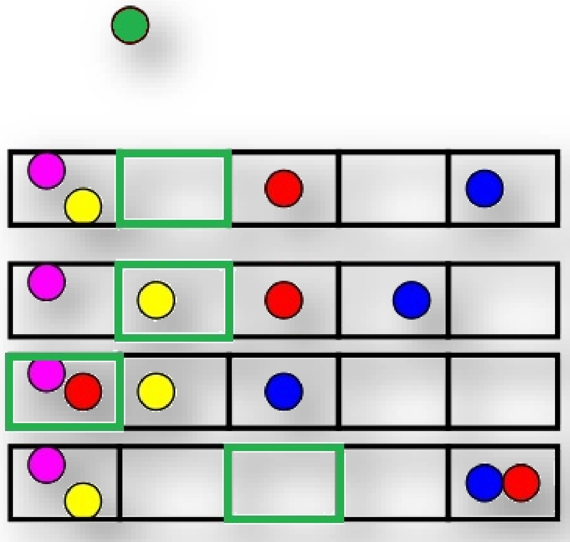
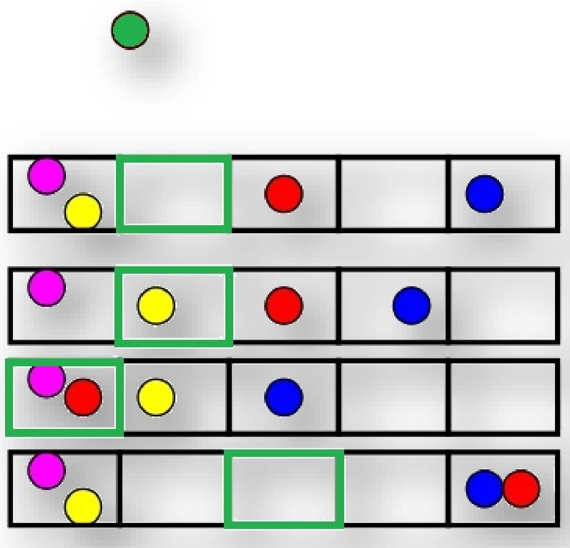
3. Keep p with probability $\frac{1}{d_p}$, o.w. repeat



Algorithm

How to output a random neighbor from $\cup\mathcal{G} = \cup_{F\in\mathcal{G}} F$

1. Choose a set $F \in \mathcal{G}$ w.p. $\propto |F|$
2. Choose a uniformly random point in F
   - Each point is picked w.p. proportional to its degree d_p
3. Keep p with probability $\frac{1}{d_p}$, o.w. repeat
   - Uniform probability



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$\boldsymbol{L} = |\mathcal{G}|$

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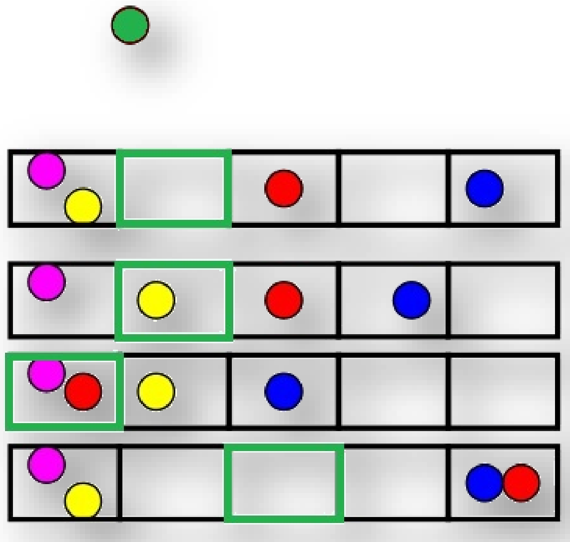

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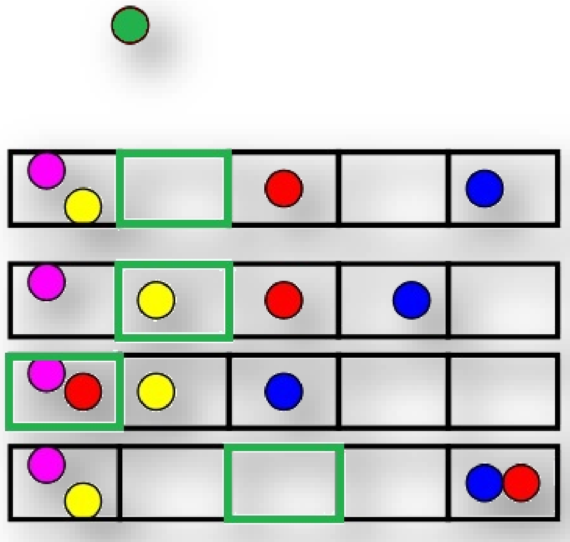
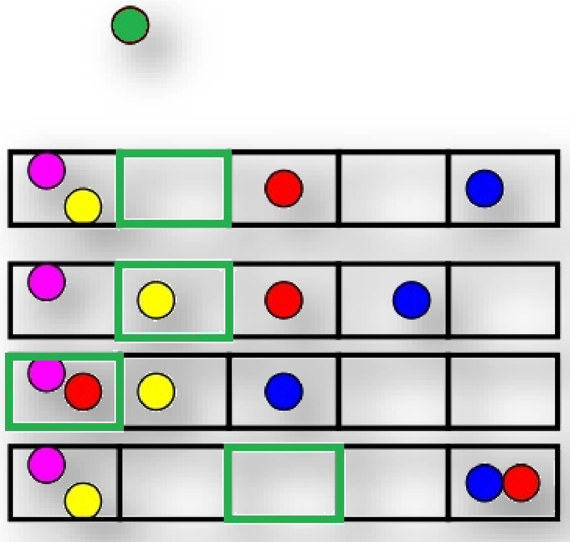
   - Might need $O(d_{max}) = O(L)$ samples
   - Total time is $O(L^2)$



Approximate the degree d_p

Sample $O(\frac{L}{d_p \cdot \epsilon^2})$ sets out of L sets in $\mathcal{G}$ to $(1+\epsilon)$-approximate the degree.

$\boldsymbol{L = |\mathcal{G}|}$



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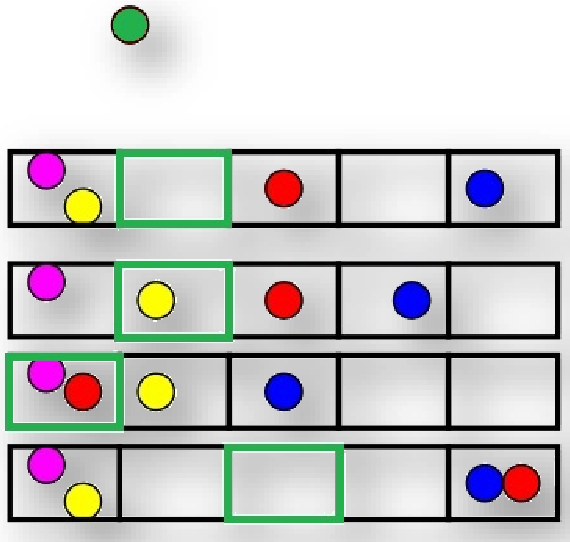

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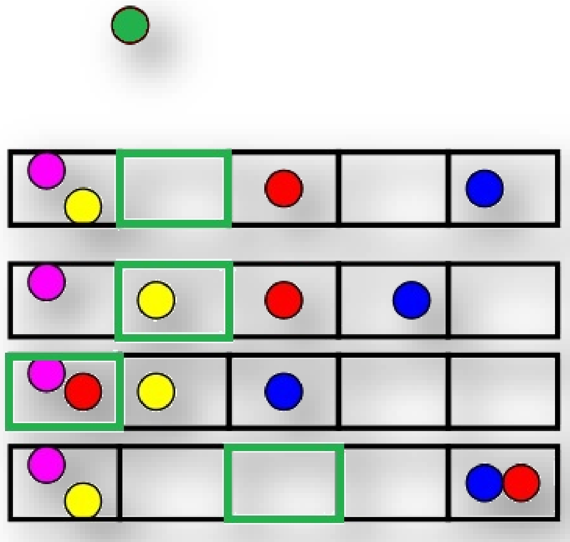
Case 1: Small degree d_p:

- More samples are required to estimate
- Reject with lower probability -> Fewer queries of this type



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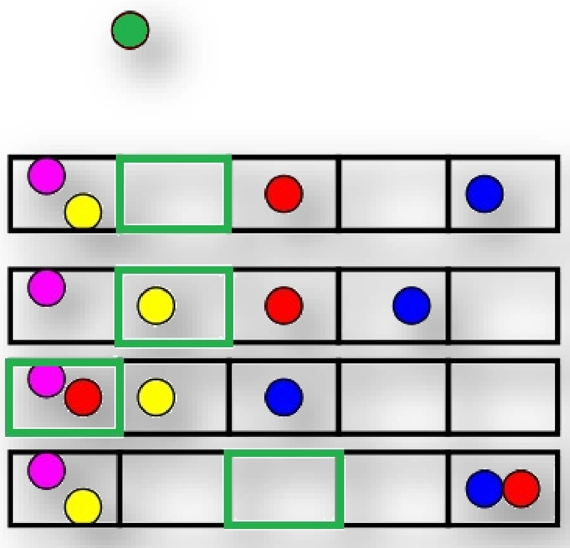
- Fewer samples are required to estimate
- Reject with higher probability -> More queries of this type



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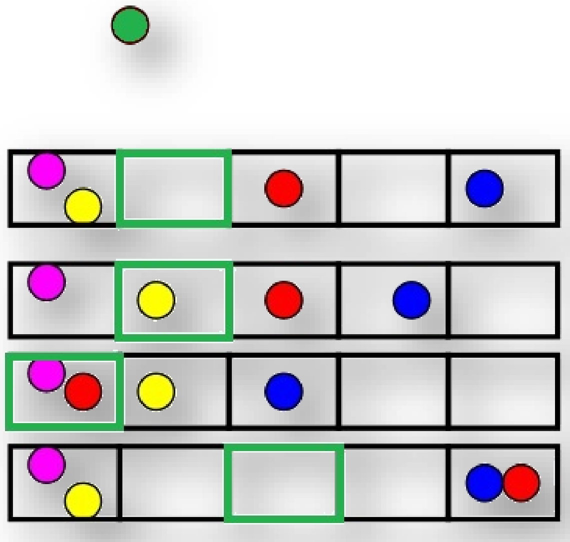

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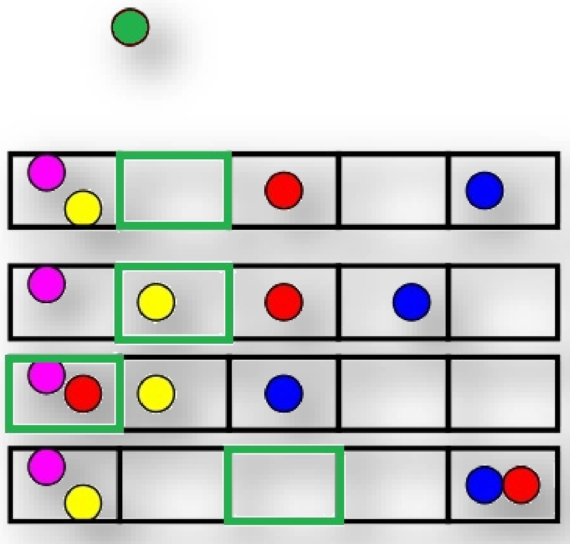

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Case 2: Large degree d_p:

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- ➢ This decreases $O(L^2)$ runtime to $\tilde{O}(L)$
- ➢ Large dependency on ϵ of the form $O(\frac{1}{\epsilon^2})$

- Nearest neighbor
- Sampling version/ fair version
- Applications
- Algorithms
  - Basic Algorithm
  - Improving the dependence on ϵ
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Improving the dependence on ϵ

From $1/\epsilon^2$ to $\log(1/\epsilon)$

Goal: A procedure that given a sample p out of the L sets in $\mathcal{G}$

- Keeps a sample p with probability $\frac{1}{d_p}$
- In time $\tilde{O}(\frac{L}{d_p})$

$L = |\mathcal{G}|$ sets

Goal: A procedure that given a sample p out of the L sets in $\mathcal{G}$

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Need to repeat $\approx d_p$ times

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Need to repeat $\approx d_p$ times

Total runtime would be $\approx d_p \cdot \tilde{O}\left(\frac{L}{d_p}\right) = \tilde{O}(L)$

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- Sample sets from $\mathcal{G}$ until you find a set F such that $p \in F$

Assuming one can check if $p \in F$ in constant time

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  - Correct except that i/L could be larger than 1
  - Keep the sample with probability $\frac{i}{\Delta \cdot L} \approx \frac{1}{\Delta \cdot d_p}$
    - Still uniform
    - Probability that $i > (\Delta L)$ is exponentially small in Δ
    - Sufficient to set $\Delta = \log \frac{1}{\epsilon}$

$E[i] = \frac{L}{d_p}$

The number of iterations increases by a factor of Δ

So far

- Get a sample uniformly at random from the union of the buckets

- $\bigcup_i B_i\left(g_{i(q)}\right)$ is roughly the neighborhood
  - Contains all points within distance r
  - Contains at most L outlier points (farther than cr)

- What about the outliers?

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Handling Outliers

Sampling from a sub-collection of sets with outliers

Preprocess: a collection $\mathcal{F}$ of subsets of a universe U

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- Whenever you see an outlier sample, ignore it and repeat.
- Runtime in the worst case: $|\mathcal{G}| \cdot m_O$

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- **Runtime of $|\mathcal{G}| + m_o$**

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Goal: Runtime of $|\mathcal{G}| + m_o$

- Implement each bucket (each set in $\mathcal{F}$) as an array

Cnt=5

↓

2, 4, 6, 9, 3

Goal: Runtime of $|\mathcal{G}| + m_o$

- Implement each bucket (each set in $\mathcal{F}$) as an array
  - Once we encounter an outlier, swap it with the last element of the array.
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Cnt=4

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Need to **(dynamically)** sample a set with probability proportional to its **active size**

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- Build a tree on with $L = |\mathcal{G}|$ leaves containing the count of the sets in $\mathcal{G}$

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- Update the counts in time $O(\log L)$

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> ➢ We see each outlier $o \in O$ at most d_o times
> ➢ Total number of times we encounter an outlier is m_o

So far

- Get a sample uniformly at random from the union of the buckets

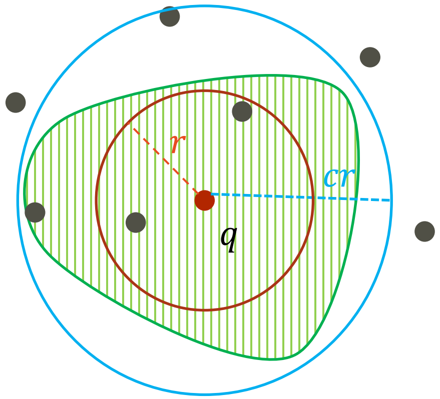
- $\cup_i B_i\left(g_{i(q)}\right)$ is roughly the neighborhood
  - Contains all points within distance r
  - Contains at most L outlier points (farther than cr)

- What about the outliers?
  - Total degree of outliers is $O(L)$
  - Get a sample in time $\tilde{O}(|\mathcal{G}| + m_o) = \tilde{O}(L + L) = \tilde{O}(L)$

Results on $(1+\epsilon)$-Approximate Fair NN

Domain	Space	Query
Exact Neighborhood $N(q,r)$	$O(S_{ANN})$	$\tilde{O}(T_{ANN} + \frac{\lvert N(q,cr)\rvert}{\lvert N(q,r)\rvert})$
Approximate Neighborhood $N(q,r) \subseteq S \subseteq N(q,cr)$	$\tilde{O}(S_{ANN})$	$\tilde{O}(T_{ANN})$

- Get a sample from the union of the buckets
- Approximate neighborhood: a set S such that $N(q,r) \subseteq S \subseteq N(q,cr)$
- Dependence on ϵ is $O(\log(\frac{1}{\epsilon}))$
- Black-box reduction



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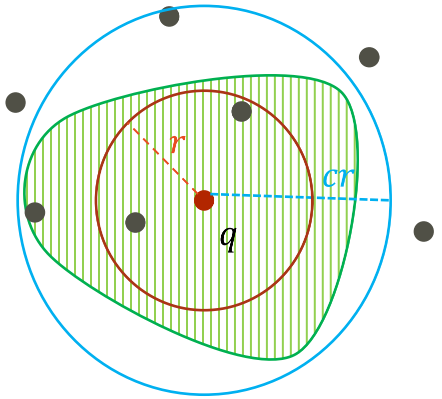

Exact Neighborhood?

- Treat the points within distance r and cr also as outliers.
- Unlucky event: we hit all the $n(q, cr)$ outliers first
- Total runtime: $\tilde{O}(|\mathcal{G}| + m_o) = \tilde{O}(L + |N(q, cr)| - |N(q, r)|) = \tilde{O}(L + |N(q, cr)|)$

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Improve to

$$T_{ANN} + \frac{|N(q,cr)|}{|N(q,r)|}$$

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Improving the dependence on the density of the neighborhood

From $T_{ANN} + |N(q, cr)|$ to $T_{ANN} + \frac{|N(q,cr)|}{|N(q,r)|}$

High Level Idea:

- Partition the elements $\cup\mathcal{G}$ **randomly** into k bins s.t.
  - Each bin gets $O(1)$ good elements, i.e., from $\cup\mathcal{G} \setminus O$
  - Each bin gets $O(\frac{|O|}{|\cup\mathcal{G}\setminus O|})$ points from the outliers
- Time will improve to $\tilde{O}(|\mathcal{G}| + m_o) = (L + \frac{|\boldsymbol{N(q,cr)}|}{|\boldsymbol{N(q,r)}|})$

More Precisely,

Preprocess:

- To partition all elements in U among k bins
  - Give each of the elements in U a random unique rank from 1 to $N = |U|$, (i.e, pick a random permutation)
  - Each set in $\mathcal{F}$ stores its elements in sorted order

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Query Time:

- Consider k bins based on the ranks,   i.e.,
  Bin i $= [\left(\frac{N}{k}\right) i\,, \left(\frac{N}{k}\right)(i+1)]$

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- Consider k bins based on the ranks, i.e.,
  Bin i $= [\left(\frac{N}{k}\right) i \, , \left(\frac{N}{k}\right)(i+1)]$
- Select one bin (almost) uniformly at random
- Get a sample from the sampled bin

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  - Each set in $\mathcal{F}$

How to choose k

- **k large:** many bins get no element from $\cup\mathcal{G}$
- **k small:** finding an element in $\cup\mathcal{G}$ that is in a particular bin takes a long time

➢ Set k roughly equal to $|\cup\mathcal{G}|$. Then each bin has roughly $O(1)$ elements from $\cup\mathcal{G}$

➢ Don't know $|\cup\mathcal{G}|$ in advance

  ➢ Count the number of distinct elements using a sketch for Distinct Elements Problem

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  ➢ Count the number of distinct elements using a sketch for Distinct Elements Problem

❑ Set $\boldsymbol{k} = \boldsymbol{n}(\boldsymbol{q}, \boldsymbol{r})$
❑ Number of outliers in a bin is at most $\boldsymbol{n}(\boldsymbol{q}, \boldsymbol{cr})/\boldsymbol{n}(\boldsymbol{q}, \boldsymbol{r})$

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Query Time:

- Consider k bins based on the ranks, i.e.,
  Bin i $= [\left(\frac{N}{k}\right) i \,, \left(\frac{N}{k}\right) (i+1)]$
- Select one bin (almost) uniformly at random
- **Get a sample from the sampled bin**

How to sample from $\cup\mathcal{G} \cap bin_i$?

- One can iterate over $F \cap Bin_i$ in time $O(\log n + |F \cap Bin_i|)$
  - Because the elements are kept sorted in F
  - And the Bin is continuous

➢Compute $|F \cap Bin_i|$ for each $F \in \mathcal{G}$

➢Build a BST on these counts, sample from them

Results on $(1+\epsilon)$-Approximate Fair NN

Domain	Space	Query
Exact Neighborhood $N(q,r)$	$O(S_{ANN})$	$\tilde{O}(T_{ANN} + \frac{\vert N(q,cr)\vert}{\vert N(q,r)\vert})$
Approximate Neighborhood $N(q,r) \subseteq S \subseteq N(q,cr)$	$\tilde{O}(S_{ANN})$	$\tilde{O}(T_{ANN})$

- S_{ANN} and T_{ANN} are the space and query time of standard ANN
- Approximate neighborhood: a set S such that $N(q,r) \subseteq S \subseteq N(q,cr)$
- Dependence on ϵ is $O(\log(\frac{1}{\epsilon}))$
- Black-box reduction
- Our approach solves a more general problem
- Experiments

Summary

- Defined NN problem with respect to fairness, i.e., the sampling variant
  - Applications of sampling NN
- How to sample from a sub-collection of sets
- Improve dependency on ϵ
- How to handle outliers
- Improve dependency on the density parameter of the neighborhood

Summary

Thanks

Questions?

Domain	Space	Query
Exact Neighborhood $N(q,r)$	$O(S_{ANN})$	$\tilde{O}(T_{ANN} + \frac{\lvert N(q,cr)\rvert}{\lvert N(q,r)\rvert})$
Approximate Neighborhood $N(q,r) \subseteq S \subseteq N(q,cr)$	$\tilde{O}(S_{ANN})$	$\tilde{O}(T_{ANN})$

- S_{ANN} and T_{ANN} are the space and query time of standard ANN
- Approximate neighborhood: a set S such that $N(q,r) \subseteq S \subseteq N(q,cr)$
- Dependence on ϵ is $\mathrm{O}(\log(\frac{1}{\epsilon}))$
- Black-box reduction
- Our approach solves a more general problem
- Experiments

Open Problem:

- Finding the optimal dependency on the density parameter